e is everywhere

From determining the compound interest on borrowed money to gauging chances at the roulette wheel in Monte Carlo, Stefanie Reichert explains that there's no way around Euler's number.

Even outside school or university, we cannot escape Euler's number. Jacob Bernoulli is credited with discovering e while thinking about matters of continuous compound interest in 1683. He realized that when the compounding period became smaller and smaller and more and more periods were considered, the amount of money would converge towards a limit that was later found to be one of the representations of e. Since then, use of Euler's number has become more widespread and now it appears in many branches of science and in everyday life.

For example, Euler's number shows up in probability theory. Imagine you are in Monte Carlo enjoying a few games of roulette, which is a Bernoulli trial process. If you place a bet on a single number, your chances are 1/37 to win that game. For 37 games, the probability that you will lose every single time is — maybe surprisingly — close to 1/e. Or, pretend you are at the theatre, where you — along with everybody else — leave your coat in the cloak room, which has one hook per guest, and receive a number. However, your coat is placed on a random hook. The probability that none of the coats are on the correct hook for a large number of guests approaches, again, 1/e. The number of practical examples is endless.

The history of e reads like the Who's Who of mathematics and physics. It all started with the discovery of the logarithm by John Napier: Euler's number is hidden deep in the many pages of the appendix tabulating natural logarithms in his 1614 work *Mirifici Logarithmorum Canonis Descriptio*. Later, when Bernoulli studied the case of continuous compound interest, he concluded that the limit must converge to a number between 2 and 3. As it turned out, this limit equals Euler's number (less commonly known as Napier's constant), and Bernoulli came up with its first approximation.

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It took a while before scholars connected the dots and realized that the base of the logarithm introduced by Napier and the limit discovered by Bernoulli were closely related and settled on a common notation. Gottfried Leibniz referred to what is now known as Euler's number as *b* in discussions with Christiaan Huygens, whereas others such as Jean-Baptiste le Rond d'Alembert preferred to use the notation *c* instead. This dispute was eventually settled when the Swiss mathematician Leonhard Euler (pictured) used the letter e in an early essay on the firing of cannons — and his choice became increasingly popular.

Similar to π, Euler's number $e \approx 2.71828$ is irrational and also transcendental — meaning it doesn't form a solution of a non-zero polynomial equation with integer coefficients. Whether e (or π) is a normal number remains to be determined. A normal number consists of a sequence of digits in which single digits between 0 and 9 occur with a frequency of 10%, whereas each pair of digits between 00 and 99 occurs with a frequency of 1%, and so on.

Euler is credited with a whole bunch of constants besides e, so one should be careful not to mix Euler's number up with Euler's constant, also called the Euler–Mascheroni constant, $\gamma \approx 0.57721$, defined as the limit of the difference between the harmonic series and the natural logarithm. The Euler–Mascheroni constant appears, for example, in the Bessel function of the second kind, and has not been proven to be irrational or transcendental. Another tricky case are Euler numbers (also known as zig or secant numbers), referring to the number of odd alternating permutations in expressions for the secant and hyperbolic secant (https://go.nature.com/2N0G3tc). To complicate things further, at least three other mathematical terminologies are in use denoting the Euler number of a finite complex, Euler primes or the Euler characteristics, a topological invariant. And in fluid dynamics, the Euler number characterizes the energy loss in a flow.

We have all encountered Euler's number in more ways than one — from natural logarithms to the definition of the exponential function, which relies on the series expansion of e discovered by Euler himself in 1748. The constant e appears practically everywhere in science: popping up in the definition of the standard normal distribution; allowing us to decompose a time-dependent signal into its frequencies via Fourier transformation; telling us how to calculate the half-life of radioactive elements; playing a crucial role in the growth of bacteria; and governing temperature-activated chemical reactions.

Not for nothing, e counts among the most important constants in mathematics and physics, along with 0, 1, i and π that all show up in Euler's identity $e^{i\pi} + 1 = 0$. It is truly a constant in everyone's life. ❐

Stefanie Reichert
Associate Editor, Nature Physics.
e-mail: stefanie.reichert@nature.com

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